Asservissement : la transformée de laplace

* Définit

le passage de :

temporel ⟶ complexe

$x(t) \longrightarrow X(p)$

$$\boxed{X(p) = L(x(t)) = \int_0^{+\infty} x(t).\, e^{-pt}\, dt}$$

* Propriétés

- Deriva : $\Omega(t) = \frac{d\theta(t)}{dt} \Rightarrow \boxed{\Omega(p) = p.\theta(p)}$
- integra : $\theta(t) = \int \Omega(t)\, dt \Rightarrow \boxed{\theta(p) = \frac{1}{p}\,\Omega(p)}$
- théorème de la valeur finale

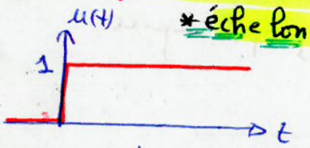
$\lim\limits_{t \to +\infty} x(t) \Rightarrow \boxed{\lim\limits_{p \to 0} p.X(p)}$

- théorème de la valeur initiale

$\lim\limits_{t \to 0} x(t) \Rightarrow \boxed{\lim\limits_{p \to +\infty} p.X(p)}$

* Transformée de laplace usuelle

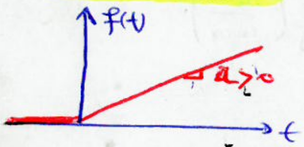
*échelon unité



$\Rightarrow \begin{cases} t \geq 0 \Rightarrow u(t) = 1 \\ t < 0 \Rightarrow u(t) = 0 \end{cases}$

$\hookrightarrow \boxed{U(p) = \frac{1}{p}}$

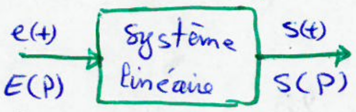
* rampe : f(t)



$\Rightarrow f(t) = a\,t.\,u(t)$

$\hookrightarrow \boxed{F(p) = \frac{a}{p^2}}$

* fonction de transfert



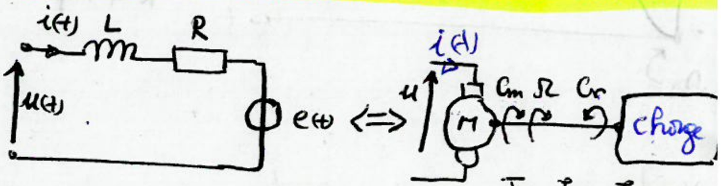
un système est linéaire s'il est représenté par une équation différentielle :

$$b_0 s(t) + b_1 \frac{ds(t)}{dt} + \dots + b_n \frac{d^n s(t)}{dt^n} = a_0 e(t) + a_1 \frac{de(t)}{dt} + \dots + a_m \frac{d^m e(t)}{dt^m}$$

$m < n$

$$\Rightarrow \boxed{H(p) = \frac{a_0 + a_1 p + \dots + a_m p^m}{b_0 + b_1 p + \dots + b_n p^n}} \Leftarrow \text{fonct de transfert}$$

Application : machine à courant continu



$J = J_m + J_{ch}$

les 4 éq de la mcc :

- $u(t) = L\frac{di(t)}{dt} + Ri(t) + e(t)$ • $e(t) = k\,\Omega(t)$
- $J\frac{d\Omega}{dt} = C_m(t) - C_r(t) - f\,\Omega(t)$ • $C_m(t) = k\,i(t)$

*** la transformée de laplace**

- $U(p) = L.p.I(p) + R\,I(p) + E(p)$ • $E(p) = k\,\Omega(p)$
- $J.p.\Omega(p) = C_m(p) - C_r(p) - f\,\Omega(p)$ • $C_m(p) = k\,I(p)$

*** fonct de transfert si $C_r = 0$**

on : $\hookrightarrow H(p) = \frac{\Omega(p)}{U(p)}$

$J p \Omega(p) + f\,\Omega(p) = C_m(p) = k\,I(p)$

$\Leftrightarrow \Omega(p)[Jp + f] = k\,I(p)$ ①

et que : $I(p)(L.p + R) = U(p) - E(p)$

$\hookrightarrow I(p) = \frac{U(p)}{Lp+R} - \frac{E(p)}{Lp+R}$

$\Leftrightarrow I(p) = \frac{U(p)}{Lp+R} - \frac{k\,\Omega(p)}{Lp+R}$ ②

① et ②

$\Rightarrow \Omega(p)\left[J.p + f + \frac{k^2}{Lp+R}\right] = \frac{U(p)\,k}{Lp+R}$

finalement :

$$\boxed{H(p) = \frac{k}{k^2 + Rf + (RJ + Lf)p + LJ\,p^2}}$$

on cherche la sortie $\Omega(p)$ si $U(t)$ est un échelon

$H(p) = \frac{\Omega(p)}{U(p)} \Rightarrow U(p) = \frac{U_0}{p}$

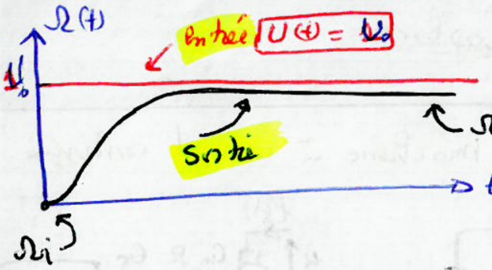
$$\Rightarrow \boxed{\Omega(p) = \frac{U_0}{p} \times \frac{k}{k^2 + Rf + (RJ + Lf)p + LJ\,p^2}}$$

- la valeur finale

$\Omega_f = \lim\limits_{p \to 0} p.\Omega(p) \Rightarrow \boxed{\Omega_f = \frac{k.U_0}{k^2 + Rf}}$

- la valeur initiale

$\Omega_i = \lim\limits_{p \to +\infty} p.\Omega(p) \Rightarrow \boxed{\Omega_i = 0}$



$$\Omega f = \frac{k\,U_0}{k^2 + Rf}$$

* **diagramme de Bode**

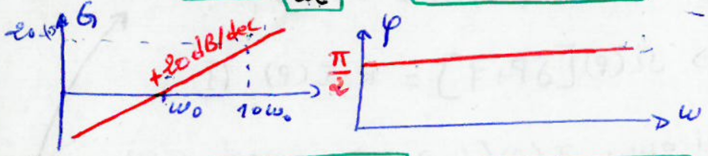
$$H(p) \xrightarrow{\;p \to j\omega\;} H(j\omega)$$

. il s'agit de le représentant de

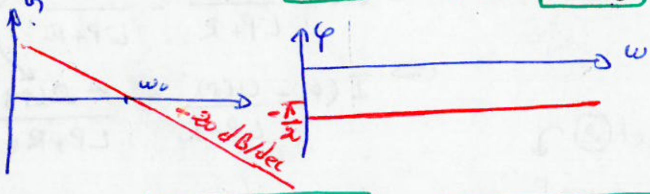
$$\begin{cases} \boxed{G(j\omega) = 20\log_{10}\left(|H(j\omega)|\right)} \\ \boxed{\varphi(j\omega) = \operatorname{Arg}\left(H(j\omega)\right)} \end{cases}$$

• Pour tracer les diagramme de Bode on se basant de :

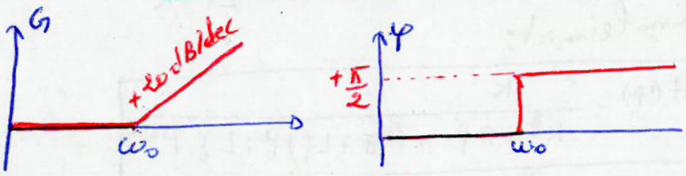
• forme ① : $\boxed{H(p) = \frac{p}{\omega_0}} \Rightarrow \boxed{H(j\omega) = j\frac{\omega}{\omega_0}}$



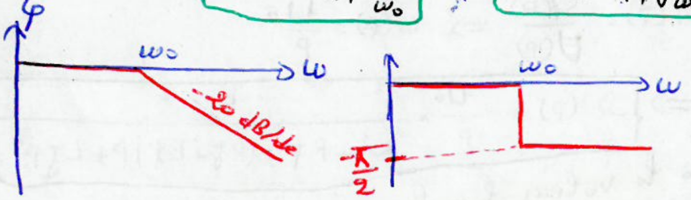
• forme ② : $\boxed{H(p) = \frac{1}{p/\omega_0}} \Rightarrow \boxed{H(j\omega) = \frac{1}{j\frac{\omega}{\omega_0}}}$



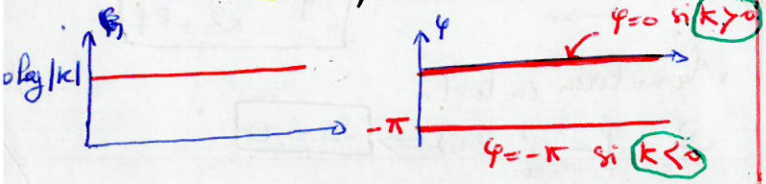
• forme ③ : $\boxed{H(p) = 1 + \frac{p}{\omega_0}} \Rightarrow \boxed{H(j\omega) = 1 + j\frac{\omega}{\omega_0}}$



• forme ④ : $\boxed{H(p) = \frac{1}{1 + \frac{p}{\omega_0}}} \Rightarrow \boxed{H(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_0}}}$



forme 5 : $H(p) = K$, $\Rightarrow H(j\omega) = K$



EX : machine à courant continu

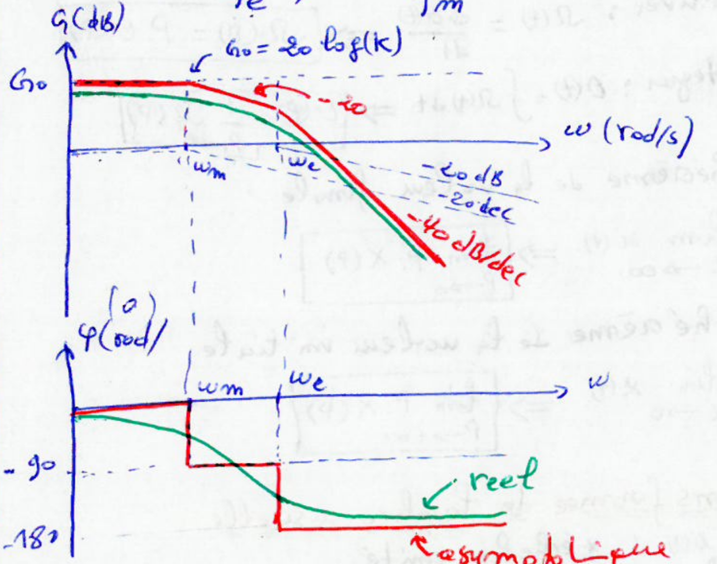
$$H(p) = \frac{K}{(1 + T_e p)(1 + T_m p)} \quad \text{avec } K > 0$$

$\hookrightarrow T_m \gg T_e\ (s)$

$$\Rightarrow H(j\omega) = \frac{K}{(1 + T_e j\omega)(1 + T_m j\omega)}$$

$$= \frac{K}{\left(1 + j\frac{\omega}{\omega_e}\right)\left(1 + j\frac{\omega}{\omega_m}\right)}$$

avec $\omega_e = \frac{1}{T_e}$, $\omega_m = \frac{1}{T_m}$ et $\omega_m \ll \omega_e$



$$\begin{cases} |H| = \dfrac{K}{\sqrt{\left[1 + \left(\frac{\omega}{\omega_e}\right)^2\right]\left[1 + \left(\frac{\omega}{\omega_m}\right)^2\right]}} \\ \varphi = -\operatorname{arctg}\left(\frac{\omega}{\omega_e}\right) - \operatorname{arctg}\left(\frac{\omega}{\omega_m}\right) \end{cases}$$

Rappel :

$$\left(1 + j\frac{\omega}{\omega_0}\right)^n \quad \text{ou} \quad \left(j\frac{\omega}{\omega_0}\right)^n$$

$$\text{Pente} = n \times 20$$

$$\varphi = n \times \frac{\pi}{2}$$

$$\arg\left(\frac{a}{b}\right) = \arg(a) - \arg(b)$$

$$\arg(a \times b) = \arg(a) + \arg(b)$$

$$\operatorname{arctg}(a) + \operatorname{arctg}(b) = \operatorname{arctg}\left(\frac{a+b}{1-ab}\right)$$